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Math Logic
Fall 2004

Proof: ($\sum$ consistent $\rightarrow \sum$ satisfiable) $\rightarrow$ ($\sum \models \alpha \rightarrow \sum \vdash \alpha$)

This Lemma is used in the final steps of the Completeness theorem, which claims the second part of the implication: $\sum \models \alpha \rightarrow \sum \vdash \alpha$. If every model of $\sum$ is a model of α, then there is a proof of α from $\sum$.

- $\sum consistent \leftrightarrow \forall\beta(\sum \nvdash (\beta \wedge \neg\beta))$
- $\sum satisfiable \leftrightarrow \exists A(A \models \sum)$
- $\sum \models \alpha \leftrightarrow \forall A(A \models \sum \rightarrow A \models \alpha)$

Now, using a proof by contradiction, we prove the lemma at hand. For a proof by contradiction in this example, we assume (a) $\sum$ consistent $\rightarrow \sum$ satisfiable, (b) $\sum \models \alpha$, and (c) $\sum \nvdash \alpha$.

(1)	$(\forall\beta(\sum \nvdash (\beta \wedge \neg\beta))) \rightarrow (\exists A(A \models \sum))$	$: (a)$
(2)	$\forall A(A \models \sum \rightarrow A \models \alpha)$	$: (b)$
(3)	$\sum \nvdash \alpha$	$: (c)$
(4)	$\forall\beta(\sum \nvdash (\beta \wedge \neg\beta))$	$: (3) tricky...$
(5)	$\forall\beta(\sum \bigcup \{\neg\alpha\} \nvdash (\beta \wedge \neg\beta))$	$: (3), (4)$
(6)	$\exists A(A \models \sum \bigcup \{\neg\alpha\})$	$: (1), (5)$
(7)	$\exists A(A \models \neg\alpha \wedge A \models \sum)$	$: (6)$
(8)	$\exists A(A \models \neg\alpha \wedge A \models \alpha)$	$: (2), (7)$

However, the last step contradicts the definition of a model, thus one of the initial assumptions must be false, namely the third one which states $\sum \nvdash \alpha$.

$$\therefore (\sum \text{ consistent} \rightarrow \sum \text{ satisfiable}) \rightarrow (\sum \models \alpha \rightarrow \sum \vdash \alpha)$$